

Generalized binomial transform applied to the divergent series

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The divergent series for a function defined through Laplace integral and the ground state energy of the quartic anharmonic oscillator to large orders are studied to test the generalized binomial transform which is the renamed version of δ -expansion proposed recently. We show that, by the use of the generalized binomial transform, the values of functions in the limit of zero of an argument is approximately computable from the series expansion around the infinity of the same argument. In the Laplace integral, we investigate the subject in detail with the aid of Mellin transform. In the anharmonic oscillator, we compute the strong coupling limit of the ground state energy and also the expansion coefficients at strong coupling from the weak coupling perturbation series. The obtained result is compared with that of the linear delta expansion.

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I. INTRODUCTION

The δ -expansion proposed in ref. [1] has been considered so far on the discretized back ground. In all applications of the method, the expansion in terms of the basic parameter has finite radius of convergence such as the strong coupling expansion in the field theoretic models on lattice and high temperature expansion in magnetic systems [2, 3]. The existence of the non-zero convergence radius played an important role in the application of the δ -expansion. The aim of this paper is to investigate whether the method is effective in asymptotic series appearing in perturbation expansion. Here we will apply the method to two models, a mathematical function defined through Laplace integral and the quantum mechanical anharmonic oscillator in which we focus on the computation of the ground state energy at strong coupling from the weak coupling perturbation theory to large perturbative orders.

Before the argument, to avoid possible confusion, we like to rename the " δ -expansion" used in [1] to the "generalized binomial transform" due to the reason to be explained below: The anharmonic oscillator can be viewed as the Euclidean 1-dimensional ϕ^4 field theory where we denote the space coordinate by q . The Hamiltonian then reads

$$H = \frac{1}{2} \left(\frac{\partial \phi}{\partial q} \right)^2 + \frac{m^2}{2} \phi^2 + \lambda \phi^4. \quad (1.1)$$

The non-linear interaction is controlled by the coupling constant λ and the un-perturbed current mass is given by m . The perturbation theory provides an expansion of any physical quantity in λ . Due to the mass dimension 3 carried by the coupling constant, expansion in λ is actually the expansion in λ/m^3 which is dimensionless. Thus it is apparent that the perturbative expansion is almost equivalent with the inverse-mass expansion. There exists a novel computational technique called "linear delta (δ) expansion", "optimized perturbation theory" or "variational perturbation theory" [4]. Also the "order dependent mapping" [5] is another adjacent method which

includes "linear delta (δ) expansion" in a specific fixing of the mapping. In anharmonic oscillator, these techniques are similar to the " δ -expansion" proposed in [1]. The linear delta expansion introduces δ as the interpolating parameter by the substitutions $m^2 \rightarrow m^2(1-\delta)$ and $\lambda \rightarrow \lambda\delta$. The Hamiltonian to start with is

$$H(\delta) = \frac{1}{2} \left(\frac{\partial \phi}{\partial q} \right)^2 + \frac{m^2}{2} \phi^2 + \delta \left(-\frac{m^2}{2} \phi^2 + \lambda \phi^4 \right). \quad (1.2)$$

Notice that $\delta = 0$ reduces the system to the free massive oscillator and $\delta = 1$ to the massless anharmonic oscillator (pure anharmonic oscillator). Linear delta expansion regards $H(0) = \frac{1}{2}(\partial\phi/\partial q)^2 + (m^2/2)\phi^2$ as the unperturbed part and expand the perturbation $\delta[-(m^2/2)\phi^2 + \lambda\phi^4]$ as the power series in δ . The result shows the perturbative result of the interpolated system with the mass $m^2(1-\delta)$ and the coupling constant $\delta\lambda$. Then, setting $\delta = 1$, it is known that nontrivial and effective estimates of physical observables in the massless limit (or the strong coupling limit) are obtained. The concerned papers are quite many and see, for example, the paper [4] and references therein. For the application of interpolating linear delta expansion on the lattice, see [6].

The " δ -expansion" proposed in ref. [1] has been derived in the similar technique. Suppose that $f(m^2)$ be given as the truncated series in $1/m^2$ to order N ,

$$f_N(m^2) = \sum_{n=0}^N a_n \left(\frac{1}{m^2} \right)^n. \quad (1.3)$$

By the re-scaling of the argument $m^2 = (1-\delta)/t$, expanding $f_N((1-\delta)/t)$ in δ to the relevant order and setting $\delta = 1$, one obtains the δ -expansion of f as the function of t (See (1.4)). This technique is first used on the lattice as a tool of dilatation of the continuum scaling region, where the argument is basically related to the lattice spacing. On the other hand, the linear delta-expansion is stemmed from the interpolation of two different systems. Though the two tools share similar features and sometimes produce same results, they differ in the underlying concept and specific details, in particular when applied to physical systems on the lattice. This is the reason of renaming the " δ -expansion" to "generalized binomial transform" or simply in short "binomial transform".

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It would be in order to review briefly the generalized binomial transform: In typical cases, the generalized binomial transform acts on the simple truncated power series (1.3) as

$$\mathcal{B}_N[f_N(m^2)] = \sum_{n=0}^N a_n \binom{N}{n} t^n. \quad (1.4)$$

That is, the coefficient a_n in original series is multiplied by the binomial factor

$$\binom{N}{n} = \frac{N!}{n!(N-n)!}, \quad (0 \leq n \leq N). \quad (1.5)$$

Thus, the binomial transform denoted by \mathcal{B}_N is dependent on the perturbative order N . For the sake of notational simplicity, we also use "bar" representing the transform,

$$\mathcal{B}_N[f_N(m^2)] := \bar{f}_N(t). \quad (1.6)$$

At first sight, one may feel difficulty in recognizing the effectivity of $\bar{f}_N(t)$ in extracting the quantities, the limit $\lim_{m \rightarrow 0} f(m^2) = f(0)$ when convergent and the critical exponent when divergent in power-like manner as $m \rightarrow 0$. Though $\bar{f}_N(t)$ is just a polynomial in t , we found in some physics models that the asymptotic behavior of $f(m^2)$ as $m \rightarrow 0$ is observable in $\bar{f}_N(t)$ at *non-large t region* and marking quantities (the limit and critical exponents) can be estimated. For instance in the square Ising model at temperature $1/\beta$, the effective region of $\mathcal{B}_N[\beta(M)] = \bar{\beta}_N(t)$ in the $N \rightarrow \infty$ is numerically assured to be $(0, 0.25)$ [2] over which the function exhibits an extremely flat plateau and the inverse critical temperature β_c is indicated at the stationary or almost stationary point of the function $\bar{\beta}_N(t)$. Here the mass argument M is composed by the magnetic susceptibility χ and the second moment μ_2 as $M = 4\chi/\mu_2$. For convergent series, it is implied that the limit $\lim_{N \rightarrow \infty} \mathcal{B}_N[f_N]$ is constant over $0 < t < t_c$ for a certain t_c and the function in the $N \rightarrow \infty$ limit has the shape like the step function with finite range of $(0, t_c)$.

Turning to the anharmonic oscillator, we deal with the perturbative expansion in powers of the coupling constant λ . Then, remembering that the ground state energy $E(m, \lambda)$ has an asymptotic expansion in λ/m^3 with alternate sign, there are two crucial differences compared to the Ising case and 2D large N vector model discussed in refs. [1–3]. The first is that the expansion parameter has fractional powers of $1/m^2$ such as $(1/m^2)^{-1/2}$, $(1/m^2)^{5/2}$ and so on. To handle these terms, we use the generalized binomial factor,

$$\binom{N}{s} = \frac{\Gamma(N+1)}{\Gamma(s+1)\Gamma(N-s+1)}, \quad (1.7)$$

where s is real or complex when convenient and necessary, and the transformation rule given by

$$\mathcal{B}_N[M^{-s}] = \binom{N}{s} t^s, \quad M = m^2. \quad (1.8)$$

As we shall see in the next section, this rule is suitable when the function $f(M)$ of interest allows Mellin transform representation.

Second, as aforementioned, the original series has zero convergence radius and it is unclear at all whether the generalized binomial transform effectively works as before. In this paper, we will demonstrate that a careful use of the principle of minimum sensitivity (PMS) [7] provides an accurate sequence of estimates to large enough orders.

This paper is organized as follows: In the second section, we apply the binomial transform to a mathematical function which allows divergent expansion such that $(1/M) - 2!(1/M)^2 + 3!(1/M)^3 - \dots$ and investigate in detail the computation of the limit $M \rightarrow 0$ from the divergent series. The technique of Mellin transform representation is introduced to make the analysis transparent. From this example, one can gain concrete feeling of how binomial transform works. In the third section, we investigate the application of the binomial transform to the anharmonic oscillator. We first review weak coupling perturbation expansion and consider its transform. The linear delta-expansion is also mentioned and the difference is explained. Explicit estimation will be worked out to the order $N = 300$. The sequence of the estimates indicates the strong evidence of the convergence to the most precise value to date, even though the region "effective" in the estimation shrinks as the order grows. Also presented is the computation of the strong coupling coefficients and compared with those from the conventional linear delta expansion. The binomial transform related to the dilatation of the region around $\lambda = \infty$ is finally investigated. The last section is devoted to the concluding remarks.

II. A LAPLACE INTEGRAL

A. Mellin transform

In the use of transformation rule (1.8) to the closed form of function, Mellin transform plays an important role. Given a function f as the argument M , the representation through Mellin transform reads

$$f(M) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} M^{-s} \varphi(s), \quad (2.1)$$

where

$$\varphi(s) = \int_0^\infty dM M^{s-1} f(M). \quad (2.2)$$

In (2.1), the integration contour in the complex s -plane is taken as the vertical one passing through $c \in \mathbb{R}$ and it is assumed that the integral (2.1) exists in a certain vertical strip including the point $(c, 0)$.

The expansion of $f(M)$ at small M is given by the deformation of the contour to the left half-plane, by which residues at supposed poles leave the required series. As well, the expansion in $1/M$ is obtained by the deformation of the contour to the right half-plane.

When the Mellin transform representation is available, the generalized binomial transform is easy to implement. We find

the result from (1.8)

$$\begin{aligned}\bar{f}(t) &= \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \mathcal{B}_N[M^{-s}] \varphi(s) \\ &= \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \frac{\Gamma(N+1)}{\Gamma(s+1)\Gamma(N-s+1)} t^s \varphi(s). \quad (2.3)\end{aligned}$$

The kernel changes from $\varphi(s)$ to $\frac{\Gamma(N+1)}{\Gamma(s+1)\Gamma(N-s+1)}\varphi(s)$. Deformation of the contour to the left half-plane gives the expansion of $\bar{f}(t)$ in $1/t$. Thus, the large M behavior of $f(M)$ corresponds to the small t behavior of $\bar{f}(t)$. If $\varphi(s)$ has single pole at $s = -L$ for positive integer L , expansion at small M has the term M^L . In contrast for $\bar{f}(t)$, the corresponding t^{-L} term is absent since then the singularity is cancelled by $1/\Gamma(s+1)$. If $\varphi(s)$ has double poles at $s = -L$, then there appears $M^L \log M$, but for $\bar{f}(t)$ just a power-like term t^L remains and the associated residue becomes

$$\frac{(-1)^{L+1}\Gamma(N+1)\Gamma(L)}{\Gamma(N+L+1)}\varphi_{-2}, \quad (2.4)$$

where the expansion $\varphi(s) = \varphi_{-2}/(s+L)^2 + \dots$ is supposed. It is crucial to note that the residue tends to vanish as $N^{-L}\varphi_{-2}$ as $N \rightarrow \infty$. Surviving term is the residue at $s = 0$ only, provided the pole is surrounded in the contour deformation.

B. Generalized binomial transform applied to divergent expansion

Let us consider the mathematical function for $M > 0$ defined through Laplace integral given by

$$f(M) = M \int_0^\infty \frac{\omega e^{-M\omega}}{1+\omega} d\omega. \quad (2.5)$$

The rotation of the integration contour on the complex ω -plane reveals that the function $f(M)$ can be analytically extended in the complex M -plane. One then finds that the origin is a branch point and the circulation around the origin creates $2\pi i M e^M$, proving $f(M)$ be a multi-valued function.

From the well known result of Mellin transform,

$$e^{-M\omega} = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} (M\omega)^{-s} \Gamma(s), \quad \Re[s] > 0, \quad (2.6)$$

we obtain the following representation,

$$f(M) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} M^{1-s} \Gamma(s) \Gamma(s-1) \Gamma(2-s), \quad (2.7)$$

where s must obey $1 < \Re[s] < 2$. The integrand has double poles at $s = 1, 0, -1, -2, \dots$ and single poles at $s = 2, 3, \dots$. By the deformation of the integration contour to the left or the right, one obtains the series expansion in M or $1/M$, respectively. Due to the doubleness of poles, the single power of the logarithm appears in expansion in M . The result reads from the residue computation that

$$f(M) = 1 + M(\log M + \gamma_E) + O(M^2 \log M). \quad (2.8)$$

We notice that $\lim_{M \rightarrow +0} f(M) = 1$ is given by the residue at $s = 1$ and the pole $s = 1$ is the first pole one encounters in the contour deformation to the left. On the other hand, $1/M$ expansion reads

$$f(M) = \frac{1!}{M} - \frac{2!}{M^2} + \frac{3!}{M^3} - \dots \quad (2.9)$$

This series is divergent and it is impossible to find the asymptotic behavior of $f(M)$ at small enough M , or more explicitly, the estimation of $f(0) = 1$. We like to show that the binomial transform converts the $1/M$ series into the series from which $f(0)$ can be approximately computable.

The operation of the binomial transform is straightforward. We find from (1.8) and (2.7) that

$$\bar{f}(t) = N! \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \frac{\Gamma(s-1)\Gamma(2-s)}{\Gamma(N-s+2)} t^{s-1}. \quad (2.10)$$

The double poles of $f(M)$ at $s = 0, -1, -2, \dots$ have turned into the single poles and the expansion around $t = \infty$ becomes an infinite series without log. The poles at $s = N+2, N+3, \dots$ have disappeared due to the appearance of $1/\Gamma(N-s+2)$. Thus, the series expansion in t becomes a polynomial to the order N . Then

$$\bar{f}(t) = \sum_{k=1}^N \binom{N}{k} k! (-t)^k + R_N(t), \quad (2.11)$$

where the function $R_N(t)$ represents the contribution from the deformed upward contour crossing at the positive real axis at some point located to the right of the largest pole $s = N+2$. In realistic physical application, one does not have complete information and suffices truncated series to the order N . Thus, we neglect the residual contribution R_N and keep only the polynomial denoted $\bar{f}_N(t)$,

$$\bar{f}_N(t) = \sum_{k=1}^N \binom{N}{k} k! (-t)^k = N! \sum_{k=1}^N \frac{(-t)^k}{(N-k)!}. \quad (2.12)$$

For large t , gathering all residues of the poles $s = 1, 0, -1, -2, \dots$, we obtain

$$\begin{aligned}\bar{f}(t) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(N+1) \cdots (N+k) t^k} \\ &= N! \sum_{k=0}^{\infty} \frac{(-1)^k}{(N+k)! t^k}. \quad (2.13)\end{aligned}$$

This is the expansion around $t = \infty$ and manifests $\bar{f}(t)$ be an entire function in the complex $1/t$ -plane. The function $\bar{f}(t)$ is single valued with no cut. It is a crucial point here that all the coefficients except leading term tend to zero when $N \rightarrow \infty$. That is, $\bar{f}(t)$ tends to a uniform function,

$$\lim_{N \rightarrow \infty} \bar{f}(t) = 1, \quad 0 < t < \infty. \quad (2.14)$$

In the contour deformation, the pole we first encounter is $s = 1$ and the residue equals to 1. This is the limit

$\bar{f}(\infty)$. The agreement of $f(0)$ and $\bar{f}(\infty)$ is not of accidental because the residues at $s = 1$ are kept equal with each other by the generalized binomial transform (by which $\Gamma(N+1)/\{\Gamma(s)\Gamma(N-s+2)\}$, which is equal to 1 at $s = 1$, is created in the integrand). The function $\bar{f}(t)$ can be written as

$$\bar{f}(t) = N! \left[(-t)^N e^{-1/t} + \sum_{k=1}^N \frac{(-t)^k}{(N-k)!} \right]. \quad (2.15)$$

One can see that the second part agrees with $\bar{f}_N(t)$. The first term has the essential singularity at $t = 0$ and this is seen only in the deformation of the contour to the left plane. The term does not allow expansion in t and corresponds to $R_N(t)$ and leads us to understand that $\bar{f}(t)$ expressed in (2.13) and (2.15) provides the exact result of generalized binomial transform of $f(M)$ (Contribution from the infinitely remote half-circle in the left-half plane disappears).

C. Reduction of corrections to the asymptotic scaling

Now, the point is whether $\bar{f}_N(t)$ to a given order N is useful to simulate the dominant or leading behavior around $t \rightarrow \infty$ of $\bar{f}(t)$. This is where the physics problems frequently arise. By the numerical study of $f_N(t)$ we find that the transformed series shows the improved behavior compared to the original truncated series of $f(M)$. But the improvement is not sufficient and the estimation of $f(0) = \bar{f}(\infty)$ is not so good even at higher orders. The failure consists in the point that the effective region of $\bar{f}(t)$ shrinks and enough scaling behavior does not emerge due to the residual influence of corrections in $\bar{f}(t) = 1 - \frac{1}{(N+1)}t^{-1} + \dots$. To suppress the correction, L th order linear differential equation is effective to subtract the corrections,

$$\prod_{i=1}^L [1 + p_i^{-1}(d/d \log t)] \bar{f}(t) = \bar{f}(\infty) + O(t^{-(L+1)}). \quad (2.16)$$

Here, p_i denotes the exponent of $\bar{f}(t)$ expanded at large t and $p_i = i$ ($i = 1, 2, 3, \dots$). We notice that the explicit expansion of $\bar{f}(t)$ at large t is not needed here. Used knowledge is just that the expansion is in the positive integer powers of $1/t$.

The left hand side of (2.16) has small correction to $\bar{f}(\infty) = 1$ at large t of order $O(t^{-(L+1)})$. Also at small t , the correction is expected to be reduced, since at large enough N the coefficient of $t^{-(L+1)}$ vanishes as $(1/N)^{L+1}$ (see (3.25)). This suggests that $\bar{f}(t) \sim 1$ to small t region when N is large. We therefore replace $\bar{f}(t)$ in (2.16) by $\bar{f}_N(t)$ which is effective for small t supposed that at some order or above $\bar{f}_N(t)$ may be a good simulation of $\bar{f}(t)$. Let us then denote

$$\psi_L = \prod_{i=1}^L [1 + p_i^{-1}(d/d \log t)] \bar{f}_N(t). \quad (2.17)$$

By the input of exact values of p_i , we can indeed obtain better behaviors: See FIG. 2 where ψ_L for $L = 0, 1, 2, 3$ are

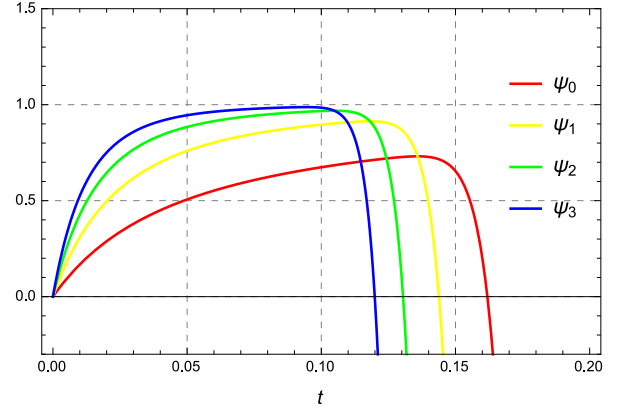


FIG. 1. 20th order plots of $\psi_L(t) = \prod_{i=1}^L [1 + p_i^{-1}(d/d \log t)] \bar{f}_N(t)$ ($L = 0, 1, 2, 3$) with the correct values $p_1 = 1$, $p_2 = 2$ and $p_3 = 3$.

TABLE I. Estimation of $f(0) = \bar{f}(\infty) = 1$ through $\psi_L(t)$ using principle of minimum sensitivity. We performed computation to 300th order, while the shown results are up to 40th. The last column labeled by " ∞ (extrapolated)" indicates the extrapolated value from the 290th and 300th results via the ansatz $\bar{f}(\infty)(1 - f_1 N^{-1})$.

N	$L = 0$	$L = 1$	$L = 2$
10	0.69276626	0.87951576	0.94485674
20	0.73101017	0.91367909	0.96853507
30	0.74556188	0.92559505	0.97580002
40	0.75337822	0.93172054	0.97928804
∞ (extrapolated)	0.78151	0.95212	0.98950
N	$L = 3$	$L = 4$	$L = 5$
10	0.97185783	0.98441767	0.99080598
20	0.98733751	0.99448001	0.99742866
30	0.99141838	0.99672795	0.99867246
40	0.99321942	0.99763006	0.99912290
∞ (extrapolated)	0.99771	0.99950	0.99989

plotted at $N = 20$. There appeared a plateau which grows flatter as the parameter numbers are increased. However, as the order N increases the plateau becomes narrower and the center moves to the origin, which is the influence of the divergent nature of $1/M$ expansion. The plateau represents, due to the successful elimination of corrections, the leading term in the $t \rightarrow \infty$, $\bar{f}(\infty) = 1$. It is natural to estimate $\bar{f}(\infty)$ on the unique top on the plateau and this protocol is called the principle of minimum sensitivity (PMS) [7]. The results of estimation using PMS are summarized in TABLE I.

The sequence of $L \geq 1$ exhibits remarkable improvement over the plane $L = 0$ sequence. As many parameters are incorporated, the accuracy becomes higher at all orders. However, the convergence issue is subtle up to the 300th order which is the highest order estimation we have performed. To settle the issue, we have done the fitting assumed ansatz

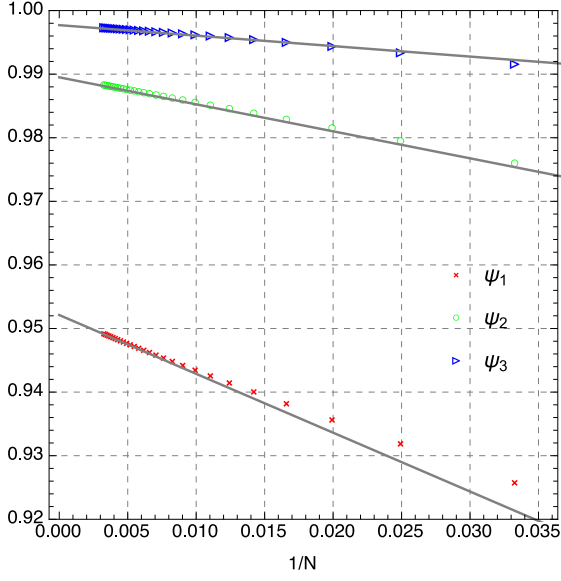


FIG. 2. Plots of estimation sequence v.s. $1/N$ for $L = 1, 2, 3$. The solid gray lines represent the fitted lines obtained via the ansatz $\bar{f}(\infty)(1 - f_1 N^{-1})$ from the results at 290th and 300th.

$\bar{f}(\infty)(1 - f_1 N^{-1})$. From the sample at $N = 290$ and 300 , which are the highest computation orders, we obtained the extrapolated values listed in TABLE I. In FIG. 2 we have shown the plots of the estimated sequence at $L = 1, 2, 3$ and the obtained fitted lines. It is confirmed that, at $L = 0 \sim 5$, the limits extrapolated do not agree with the exact value $\bar{f}(\infty) = 1$, though larger L provides better and accurate approximation. The origin of this discrepancy is again the divergent nature of $f(M)$ in $1/M$ expansion, which reflects the narrowness and movement to the origin of plateau and the top of the plateau fails to attain to the height of $\bar{f}(\infty) = 1$.

The features described so far can be explained analytically as follows: First let us consider the asymptotic behavior of the PMS solution t^* in the $N \rightarrow \infty$ limit. From numerical analysis, we find that t^* decreases as $t^* \sim c_L/N$ with c_L constant. The values of c_L are $c_0 \sim 3.4855$, $c_1 \sim 3.4252$, $c_2 \sim 3.3698$ and so on. Though these values are obtained at each L with respective highest-order (300th) values of t^* , they are in fact dependent on N . Since the behaviors of the sequences of $t^* \times N$ to the order $N = 300$ are monotonically increasing with N for all L examined, the values of c_L indicated in the $N \rightarrow \infty$ limit would be slightly larger than the above values. We can actually infer the value of true c_L to be identified only in the $N \rightarrow \infty$ limit from (2.15). For example consider the case $L = 0$. The residual part R_N is given by

$$R_N = \bar{f}(t) - \bar{f}_N(t) = N!(-t)^N e^{-1/t}. \quad (2.18)$$

Substituting the ansatz $t^* = c_0/N$ into above, we obtain $R_N = N!(-c_0/N)^N e^{-N/c_0}$ and at large enough N , from Stirling's formula, $R_N = \sqrt{2\pi N}(-c_0/e^{(1+1/c_0)})^N$. Thus, if $R_N(t^*) \rightarrow 0$ as $N \rightarrow \infty$, the condition $c_0/e^{(1+1/c_0)} < 1$ is deduced. The maximally allowed value of $c_0 = c_{0,max}$ is then found as the solution of $\log c_{0,max} = 1 + 1/c_{0,max}$, giv-

ing $c_{0,max} = 3.591121476668622 \dots$. For $L \geq 1$, the same analysis can be carried through and the result of upper limit of c_L is found to be independent of L . Thus, we conclude

$$c_{L,max} = 3.591121476668622 \dots. \quad (2.19)$$

The values of c_L for $L = 1 \sim 5$ obtained at $N = 300$ are all under and close to the above limit. Now as mentioned before, the estimated c_L grows with the order and surely tends to the value very close to or exact $c_{L,max}$ at $L = 0, 1, 2, \dots, 5$. We hence assume that estimated c_L converges to $c_{L,max}$ and compute the limit of the sequence of \bar{f}^* . First of all, we note that $\bar{f}(t)$ can be used in this study instead of \bar{f}_N itself since $R_N \rightarrow 0$ ($N \rightarrow \infty$) is assured. Then, substituting $t = c_L/N$ into $\psi_L = N! \sum_{n=0}^{\infty} (-1)^n / \{(N+n)! t^n\}$ and expanding the result in $1/N$, we obtain

$$\psi_L(t^*) = 1 - \frac{1}{(1 + c_L)^L} + O(1/N). \quad (2.20)$$

Substitution of $c_{L,max}$ into c_L produces

$$\lim_{N \rightarrow \infty} \psi_L(t^*) = 1 - \frac{1}{(1 + 3.591121476668622 \dots)^L}. \quad (2.21)$$

One finds the above result agrees with the corresponding result indicated by " ∞ (extrapolated)" in the last row in TABLE I.

The use of the exact values of the exponent is possible only when we know what values they are. The realistic physical situation in field theoretic and statistical models, the exponents are not so simple and predictable, of course. In this case, one approach is to resort to extended principle of minimum sensitivity, where the exponents are fixed as to make the higher order derivatives of ψ_L be zero at the estimation point t^* [2–4]. In the present example, however, the approach fails. It is because the higher order derivatives themselves do not reach enough scaling behaviors.

D. Estimation via Padé approximant

As the second approach, we attempt another extrapolation scheme by Padé method. Padé approximants are the rational functions constructed from the series $\bar{f}_N(t)$. It should be remind here that for the estimation of $f(0) = \bar{f}(\infty)$, the best Padé approximants among possible rational functions is the diagonal one since we can take the limit $t \rightarrow \infty$ and the result directly provides the estimation of $\bar{f}(\infty) = 1$. To define the protocol clearly, let us denote the Padé approximant of $N = \rho + \tau$ decomposition as $\bar{f}_N[\rho/\tau]$. Here ρ and τ denote the degrees of the numerator and denominator of $\bar{f}_N[\rho/\tau]$, respectively. The estimate via diagonal approximant is defined by

$$\bar{f}(\infty) = \lim_{t \rightarrow \infty} \bar{f}_N[\rho/\rho], \quad \rho = N/2. \quad (2.22)$$

For example, at $N = 10$,

$$\bar{f}_N[5/5] = \frac{10t + 160t^2 + 1470t^3 + 6960t^4 + 15240t^5}{1 + 25t + 300t^2 + 2100t^3 + 8400t^4 + 15120t^5} \quad (2.23)$$

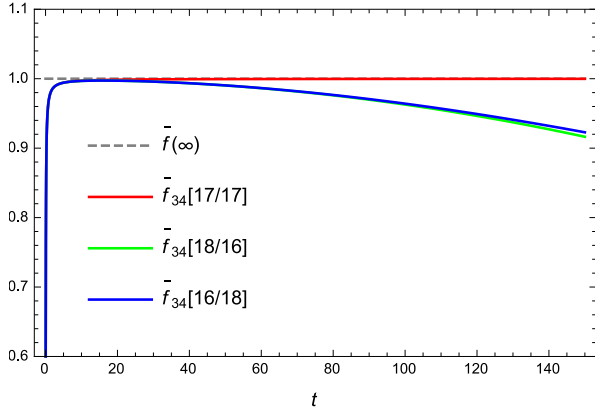


FIG. 3. Padé apprximants $\bar{f}_{34}[17/17]$, $\bar{f}_{34}[18/16]$ and $\bar{f}_{34}[16/18]$.

TABLE II. Estimation of $f(0) = \bar{f}(\infty) = 1$ using diagonal Padé approximants of $f_N(M)$ and $\bar{f}_N(t)$.

N	$f_N(M)$	$\bar{f}_N(t)$
10	0.8333333	1.0079365
20	0.9090909	0.9999891749
30	0.9375000	1.00000001289
40	0.9523809	0.9999999998549
50	0.9615385	1.0000000000000158

and

$$\lim_{t \rightarrow \infty} \bar{f}_N[5/5] = \frac{127}{126} = 1.0079365079 \dots \quad (2.24)$$

The same method is also used for $f_N(M)$. We estimated in the cases $(\rho, \tau) = (5, 5), (10, 10), (15, 15), (20, 20), (25, 25)$ for both $f_N(M)$ and $\bar{f}_N(t)$. The result is shown in TABLE II. It is clearly seen that for $f_N(M)$ the sequence is monotonically increasing and shows tendency of approaching to 1. Actually, we find from numerical work that the estimate at N th order is given by $N/(N+2) = 1 - 2/N + \dots$. The convergence speed is thus slow. As for $\bar{f}_N(t)$, the convergence tendency is strongly exhibited and in particular the accuracy is excellent. We note that the sequence here shows small oscillation with the minimum period. At $N = 2 + 4K$ ($K = 0, 1, 2, \dots$), the sequence approaches to 1 from above and at $N = 4 + 4K$ from below. In each sub-sequences, the error is exponentially small with the N dependence roughly found to be $\log_e |f_N^* - 1| \sim 3.4 - 0.693 \times N$ for both sub-sequences (f_N^* denotes the estimate at order N).

The reliability of results through diagonal approximants becomes solid when the near diagonal ones, $\bar{f}_N[\rho/\tau]$ with $|\rho - \tau| = 1$ or 2, show broad plateaus. We have observed from orders $N \sim 30$ or larger, $\bar{f}_N[\frac{N}{2} + 1/\frac{N}{2} - 1]$ and $\bar{f}_N[\frac{N}{2} - 1/\frac{N}{2} + 1]$ for even N exhibit large plateaus. See the diagonal and near-diagonal Padé approximants in the plot (FIG. 3). The reference values from the near diagonal approximants are obtained by the stationary values (local maximum

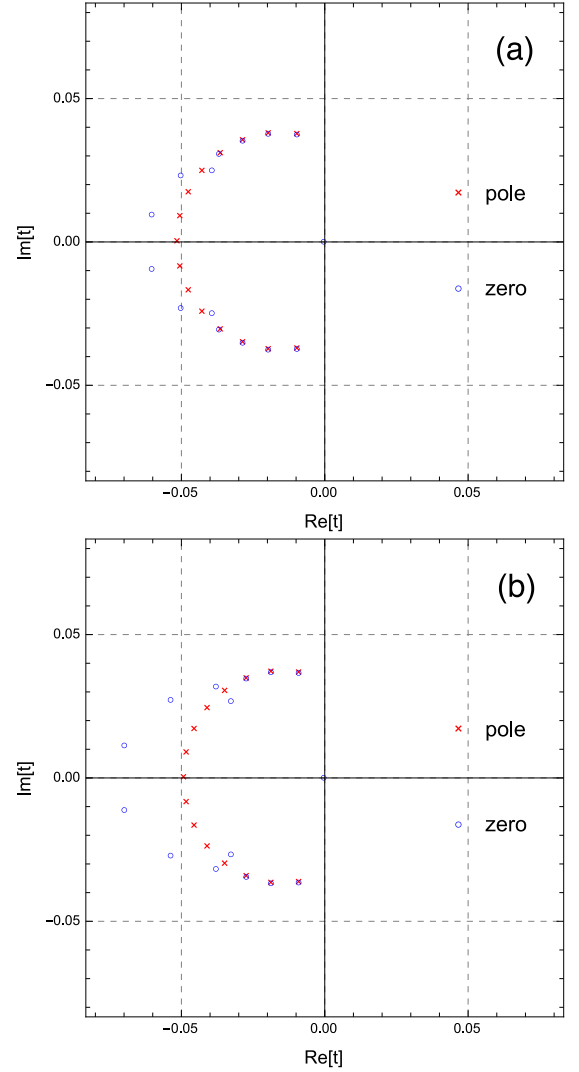


FIG. 4. Zeros and poles of Padé apprximants (a) $\psi_0[15/15] = \bar{f}_{30}[15/15]$ and (b) $\psi_1[15/15]$.

in these cases) of $\bar{f}_N[\frac{N}{2} + 1/\frac{N}{2} - 1]$ and $\bar{f}_N[\frac{N}{2} - 1/\frac{N}{2} + 1]$. At $N = 34$, they are $0.9973217 \dots$ (at $t = 15.79656 \dots$) and $0.9973225 \dots$ (at $t = 15.80567 \dots$), respectively. These values are similar in accuracy to the estimates at $L = 4$ case presented before.

At first sight, one might think that diagonal Padé approximants of $\psi_L = \prod_{i=1}^L [1 + p_i^{-1}(d/d \log t)] \bar{f}_N(t)$ would be more suitable for $L = 1, 2, 3, \dots$. Produced result are indeed accurate but not better than ψ_0 . The reason may be found by the enumeration of zeros and poles of the diagonal Padé approximants. The distribution of zeros and poles at $N = 30$ are depicted in FIG. 4. We see that for $\psi_0[15/15] = \bar{f}_N[15/15]$, the 8 poles in the left half-plane are approximately cancelled by the zeros nearby them. On the other hand for $\psi_1[15/15] = \{[1 + p_1^{-1}(d/d \log t)] \bar{f}_N\}[15/15]$, the approximate cancellation occurs for 6 pairs. Since the existence of bare poles would affect the behavior of diagonal Padé approximants on the positive real axis, it is better when the number

of bare poles are small and located in remote place (the locations of poles of $\psi_1[15/15]$ are slightly inside of those of $\psi_0[15/15]$). We can thus roughly understand why $\psi_0[15/15]$ provides better estimation. Padé approximants of the same construction for $f_N(M)$, $\prod_{i=1}^L [1 - p_i^{-1}(d/d \log M)] f_N(M)$, proves improvement for larger L . Actually, we have obtained analytic results of N th order estimate inferred from numerical study that $N(N+6)/\{(N+2)(N+4)\}$ and $N(N^2+12N+44)/\{(N+2)(N+4)(N+6)\}$ for $L=1$ and 2 , respectively. The correction to 1 is respectively $O(1/N^2)$ and $O(1/N^3)$.

We now conclude that diagonal Padé approximants of \bar{f}_N yield accurate convergent sequence of estimate.

III. ANHARMONIC OSCILLATOR

A. Perturbative expansion of the ground state energy

The perturbative ground state energy $E(m, \lambda)$ is given in the form

$$E(m, \lambda) = m \sum_{n=0}^{\infty} a_n \left(\frac{\lambda}{m^3} \right)^n. \quad (3.1)$$

The coefficient a_n can be computed from the recursion technique due to Bender and Wu [8]. For instance, the first several coefficients read

$$a_0 = \frac{1}{2}, \quad a_1 = \frac{3}{4}, \quad a_2 = -\frac{21}{8}, \quad a_3 = \frac{333}{16}. \quad (3.2)$$

We have generated the first 300 coefficients exactly and use the result in the following studies.

It was shown in ref. [8] that the coefficient grows with the order n as

$$a_n \sim -\frac{\sqrt{6}}{\pi^{3/2}} (-3)^n \Gamma(n+1/2), \quad (3.3)$$

and tells us the alternate nature with zero convergence radius. In this paper, we deal with the truncated series to the order N written as

$$E_N(m, \lambda) = m \sum_{n=0}^N a_n \left(\frac{\lambda}{m^3} \right)^n. \quad (3.4)$$

The perturbative truncation order means the number of included terms and it is matched to the order N involved in the generalized binomial factor (1.7).

B. Binomial transform and linear delta expansion

Corresponding dilatation by the re-scaling $m^2 = (1-\delta)/t$, we describe the perturbative series in terms of x defined by

$$x = \frac{\lambda^{2/3}}{m^2}. \quad (3.5)$$

The perturbative expansion is not a simple series expansion with positive integer powers but a singular expansion with fractional powers such as $x^{(3n-1)/2}$ ($n = 0, 1, 2, \dots$). With respect to such a fractional power, the binomial transform is defined with (1.7) as

$$E_N(x, \lambda) = \lambda^{1/3} \sum_{n=0}^N a_n x^{(3n-1)/2} \rightarrow \lambda^{1/3} \sum_{n=0}^N \bar{a}_n t^{(3n-1)/2}, \quad (3.6)$$

where the coefficient \bar{a}_n is given by

$$\begin{aligned} \bar{a}_n &= a_n \binom{N}{\frac{3n-1}{2}} \\ &= a_n \frac{\Gamma(N+1)}{\Gamma(\frac{3n-1}{2}+1) \Gamma(N-\frac{3n-1}{2}+1)}. \end{aligned} \quad (3.7)$$

That is, we obtain

$$\bar{E}_N(t) = \lambda^{1/3} \sum_{n=0}^N \bar{a}_n t^{(3n-1)/2}. \quad (3.8)$$

The generalized binomial transform possesses a few characteristic features which differ from the linear delta expansion as below: The first is that the factor $1/\Gamma(N-\frac{3n-1}{2}+1)$ becomes zero for some values of N (≥ 3) and n . It vanishes for $(N, n) = (3, 3), (5, 5), (6, 5), (7, 7), (8, 7), (9, 7), (9, 9)$ and so on. This leads that a subset of terms in the original expansion is eliminated. Second, the factor takes negative values for various sets of (N, n) such as $(N, n) = (4, 4), (6, 6), (7, 6), (8, 8), (9, 8)$ and so on. The negative binomial factor changes the sign of the coefficients and rigorous alternativeness is slightly broken. The original series is disturbed in this manner.

On the contrast, the linear delta-expansion does not change the sign. Some explanation would be needed here: Let us denote the result of linear delta-expansion be

$$E_{LDE, N}(m) = \lambda^{1/3} \sum_{n=1}^N a_n C_{N, n} (\lambda^{2/3}/m^2)^{(3n-1)/2}. \quad (3.9)$$

Here remind that the factor $C_{N, n}$ representing the modification comes from the term $m(\lambda/m^3)^n$ through the linear delta-expansion. One can obtain $C_{N, n}$ from the plain perturbative series (3.1) by the shifts $\lambda \rightarrow \lambda\delta$ and $m^2 \rightarrow m^2(1-\delta)$. Then, $m(\lambda/m^3)^n \rightarrow m(\lambda/m^3)^n \delta^n (1-\delta)^{-(3n-1)/2}$. The expansion of $\delta^n (1-\delta)^{-(3n-1)/2}$ in δ to the order N and setting $\delta = 1$ gives $C_{N, n}$. For example at $n=0$, we shall expand such that $(1-\delta)^{1/2} = 1 - \frac{1}{2}\delta - \sum_{k=1}^N \frac{(2k-1)!}{2^{2k-1} k! (k-1)!} \delta^k$. Then, putting $\delta = 1$, the resulting series sums to give $C_{N, 0} = (2N)!/\{2^{2N}(N!)^2\}$. For general N , $C_{N, n}$ is obtained explicitly as [9]

$$C_{N, n} = \binom{N + \frac{n-1}{2}}{\frac{3n-1}{2}} = \frac{\Gamma(N + \frac{n+1}{2})}{\Gamma(\frac{3n+1}{2}) \Gamma(N - n + 1)}. \quad (3.10)$$

As may be clear from the above procedure, the result ensures that $C_{N, n} > 0$ at any finite order N for all $n = 0, 1, 2, \dots, N$.

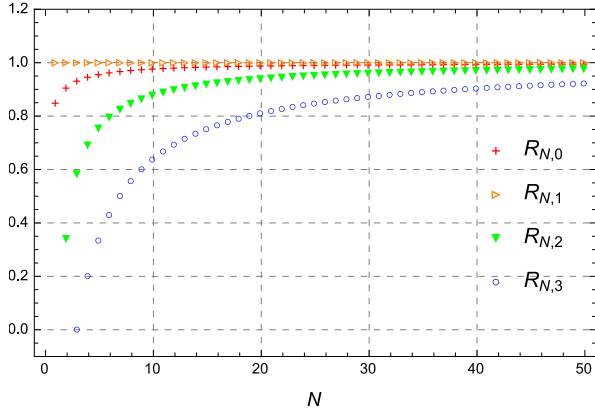


FIG. 5. Ratio plot of the leading and next-to-the leading order coefficients $R_{N,n} = \binom{N}{(3n-1)/2} / C_{N,n}$ for $n = 0, 1, 2, 3$.

We note that the factor $C_{N,n}$ is a rational number. On the other hand, $\binom{N}{(3n-1)/2}$ includes π for odd n .

For further quantitative comparison, we have plotted the ratio $R_{N,n} = \binom{N}{(3n-1)/2} / C_{N,n}$ for $n = 0, 1, 2$ and 3 against N in FIG. 5. The ratio converges to unity in the $N \rightarrow \infty$ limit as $R_{N,n} = 1 - (n-1)(3n-1)/N + O(N^{-2})$, but the difference is not negligible at finite orders except for $n = 1$ ($R_{N,1} = 1$ to all orders).

The convergence in the linear delta-expansion is proved in ref. [10]. As for the generalized binomial transform method, the proof is not obtained. However, large order numerical study provides convincing affirmative result on the convergence issue under PMS protocol by the comparison with the results of sequence in the linear delta-expansion.

C. Computation of the ground state energy

We now use $\bar{E}_N(t)$ to estimate the massless limit (or the strong coupling limit) of the ground state energy,

$$\lim_{m \rightarrow 0} E(m, \lambda) = \mathcal{E} \lambda^{1/3}, \quad (3.11)$$

where \mathcal{E} is given by Vinette and Cizek [11] to extreme accuracy [12],

$$\mathcal{E} = 0.6679862591557771082709620169198601994304049369840604559766608. \quad (3.12)$$

Before explicit computation, let us see how binomial transformed energy behaves against t . FIG. 6 shows the plot of $\bar{E}_N(t)$ at $N = 10, 20$ and 30 . It is explicitly shown that $\bar{E}_N(t)$ clearly signals the correct value already at rather small order around $N = 10$. The value to be identified as the estimate of \mathcal{E} in the plotted curves are implied by the plateaus. The width of the plateau shrinks as the order grows and this feature reflects the asymptotic nature of the original perturbative series.

We notice then the problem pointed out by Neveu in [13] that the plateau exhibits quite weak oscillation with tiny amplitudes. The oscillation may be embarrassing indeed, since

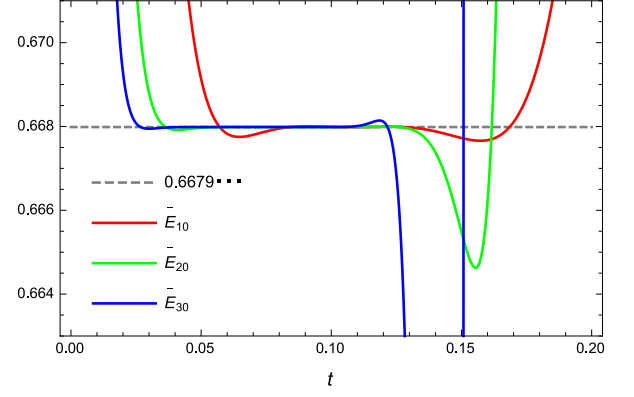


FIG. 6. Plot of $\bar{E}_N(t)$ with $\lambda = 1$ at $N = 10, 20$ and 30 .

it leads to the non-uniqueness of the stationary solution under PMS protocol. In ref. [4], Kneur, Neveu and Pinto proposed an interesting prescription to terminate this oscillation by introducing additional parameters in the linear delta expansion. Their idea is to generalize the simple prescription $m^2 \rightarrow m^2(1 - \delta)$ to the one involving more parameters such as $m^2 \rightarrow m^2(1 - \delta)(1 + (a - 1)\delta + \sum_{n=1} b_n \delta^{n+1})$ [14]. For example, at the second order, modification is to use the shift $m^2 \rightarrow m^2(1 - \delta)(1 + (a - 1)\delta)$ and expand δ as in the conventional manner. It is possible to adjust a such that only single real-valued solution, the solution satisfying $(\partial/\partial m^2)E_{LDE,N} = (\partial/\partial m^2)^2 E_{LDE,N} = 0$ exists. At the third order, they found it suffice to use $m^2 \rightarrow m^2(1 - \delta)(1 + (a - 1)\delta + b_1 \delta^2)$ and seek the unique solution obeying $(\partial/\partial m^2)E_{LDE,N} = (\partial/\partial m^2)^2 E_{LDE,N} = (\partial/\partial m^2)^3 E_{LDE,N} = 0$ under the adjustment of a and b_1 . The result was successful at low orders but turned out to getting worse at higher orders [4].

We like to remark on this problem that, without introducing additional parameters, even many oscillations occur and many candidates appear, the best optimal estimation point can be detected by carefully observing the derivatives of \bar{E}_N ; See FIG. 7 where the first order derivative $\bar{E}_N^{(1)} = (\partial/\partial \log t) \bar{E}_N$ is plotted at $N = 23$ and 50 . Seeing the plot, we find that there exists a narrow region within the plateau that the first order derivative is oscillating with smallest amplitude. With the increase of the order, the oscillatory wave becomes dense and a new oscillation wave seems to be born from the region, as signaled by the smallest amplitude of the first derivative. From this observation, we pose an assumption that the "center" of the set of zeros be optimal as the estimation point. In the case shown in FIG. 7(a), it is natural that the point indicated by the blue circle is optimal among other stationary points. In the case shown in FIG. 7(b) on the other hand, the point indicated by the red filled circle exhibits the tendency in next few orders that it goes down and across the horizontal axis, creating a new stationary point. We therefore consider the red-marked point should be considered as the optimal estimation point, even though the first derivative is not zero at the point (Note however that the value of the first derivative is extremely small there in magnitude).

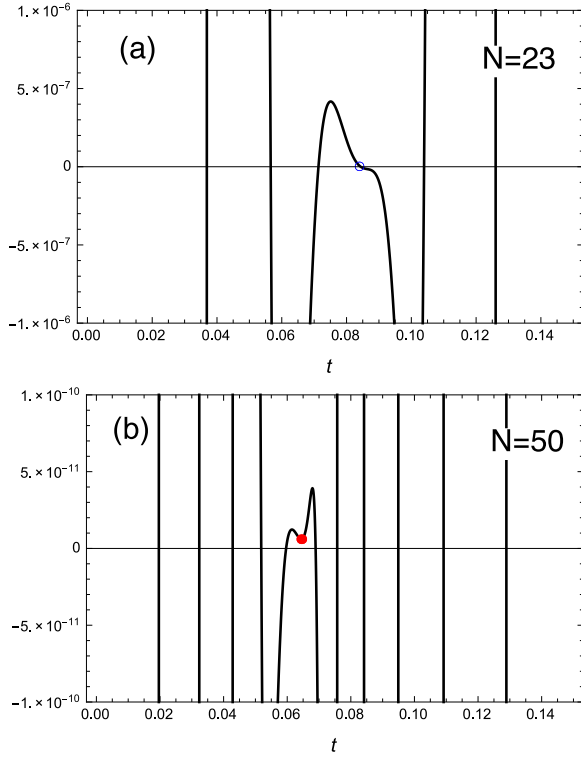


FIG. 7. First order derivatives $\bar{E}_N^{(1)}(t)$ at $N = 23$ and 50. Within the plotted range, $\bar{E}_{23}^{(1)}$ has 6 solutions and $\bar{E}_{50}^{(1)}$ has 11 solutions. Optimal stationary or almost stationary points are indicated by the blue circle (23th) and filled red blob (50th) for the estimation of \mathcal{E} . At $N = 23$, the point is at $\bar{E}_N^{(1)} = 0$ and $|\bar{E}_N^{(2)}| \ll 1$. At $N = 50$, the point is at $|\bar{E}_N^{(1)}| \ll 1$ and $\bar{E}_N^{(2)} = 0$. The later point corresponds to the reflection point with very small gradient.

It is interesting to consider the complex extension of $\bar{E}_N(t)$ denoted as $\bar{E}_N(z)$ ($z \in \mathbb{C}$) where $z = t^{3/2}$. Numerically solving $\bar{E}_N^{(1)}(z) = 0$ at $N = 50$, we have plotted the solutions in the z -plane with blue circles in FIG. 8. Red filled circles indicate the solutions of $\bar{E}_N^{(2)}(z) = [(\partial/\partial \log t)^2 \bar{E}_N]_{t \rightarrow z} = 0$. Now, the point is that there exists a small area in which the arc-shaped sequence of complex zeros and the set of real zeros on the positive real axis are intersected. As the order increases, the numbers of stationary points in each sets increase and the intersection area becomes a dense set of zeros, which we call the center of zeros. The function is smoothest there and the amplitude is smallest. The two points indicated in FIG. 7(a),(b) are located at this intersection area. The red filled circle indicated by the arrow in FIG. 8 is the red filled circle plotted in FIG. 7(b).

These observations help us handling PMS in the complicated proliferation of stationary or almost stationary points. To summarize, pick out the point in the center of zeros satisfying either (i) $\bar{E}_N^{(1)}(t) = 0$ with $|\bar{E}_N^{(2)}(t)| \ll 1$ or (ii) $\bar{E}_N^{(2)}(t) = 0$ with $|\bar{E}_N^{(1)}(t)| \ll 1$. This prescription may be regarded as a variant of the PMS criterion and we continue using the term PMS in what follows. Under the above crite-

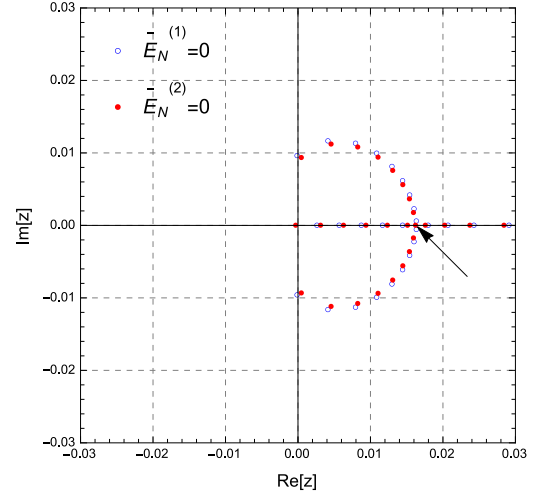


FIG. 8. The plot of zeros of the first and second order derivatives $\bar{E}_N^{(i)}(z)$ ($i = 1, 2$) at $N = 50$ in the plane $z = t^{3/2} \in \mathbb{C}$. The blue circles indicate zeros of $\bar{E}_N^{(1)}(z)$ and red filled circles zeros of $\bar{E}_N^{(2)}(z)$. We take zero of $\bar{E}_N^{(2)}(z)$ indicated by the arrow as the best optimal solution. This solution corresponds to the filled circle shown in FIG. 7(b).

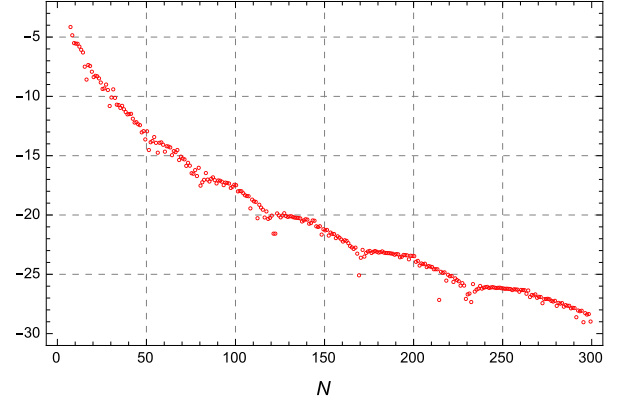


FIG. 9. Estimates of \mathcal{E} up to 300th orders. The vertical axis indicates $\log_{10} |E^* - \mathcal{E}|$ where E^* means the estimate.

ria, we have estimated \mathcal{E} to 300th orders. The result of the estimation is plotted in FIG. 9 where the vertical axis labels $\log_{10} |E^* - \mathcal{E}|$.

We observe the expected growth of the accuracy with the orders. Due to the oscillation property of $\bar{E}_N(t)$, there is a periodic pattern and the length of the period becomes longer as the order increases. In the same time, the rate of accuracy growing becomes gradually slow down, though there seems to be no limit of approaching to \mathcal{E} .

The effective region of $\bar{E}_N(t)$ shrinks as the order increases. This is already seen in FIGs. 2, 3 and 4. Accordingly, the estimation point moves to the origin with the order. The value of the estimation point t^* is plotted in FIG. 10. The precise fitting of the data is not allowed since the distribution of data is somewhat complicated with periodic structure. We just remark that, from estimates from $N = 270 \sim 300$ where the

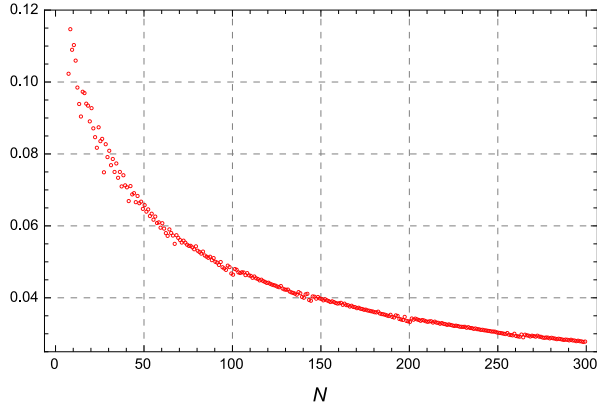


FIG. 10. Plots of the value of $t = t^*$ at which \mathcal{E} is estimated.

data are rather steady, t^* tends to zero roughly like $\sim N^{-0.56}$.

D. Comparison with the linear delta expansion

In this section, we compare the result obtained in the generalized binomial transform with the one obtained in the linear delta-expansion. The computation of the ground state energy in linear delta-expansion has been already done by Janke and Kleinert up to 251th order [15] (The expansion technique is called the variational perturbation theory). Let us first explain the work with focusing on the related part.

The PMS criterion works more clearly in the linear delta-expansion. This is understood by plotting the function $E_{LDE,N}(m)$ (see (3.9)). Omitting the graph plots, we note that the best estimation point given as the stationary point or the inflection point is always the one *at the largest value of λ/m^3* , since the oscillation amplitude becomes smallest there. It is interesting to see the distribution of zeros in the complex extension of the first order derivative $E_{LDE,N}^{(1)}(z) = [(\partial/\partial \log m^{-2})E_{LDE,N}]|_{\lambda/m^3 \rightarrow z}$ where $z = \lambda/m^3$. From the plot shown in FIG. 11, we find that the intersection of the zero point set on the positive real axis and the set extending in arc-form on the right half plane occurs at the largest real zero (The point indicated by the arrow in FIG. 11). Thus, also in the linear delta expansion, the estimation point lies on the intersection of the two sets.

The estimation result at the largest stationary point is plotted in FIG. 12 and the numerical results in both schemes (linear delta and binomial) are tabulated in TABLE III. In ref. [15], the highest order studied is 251th and the result is quoted as $\mathcal{E} = 0.66798625915577710827096$. In the result of linear delta expansion we have re-visited, we have obtained $\mathcal{E} = 0.6679862591557771082709576 \dots$ which agrees with that.

In FIG. 12, we have also plotted the results in generalized binomial transform for the sake of the comparison. At low orders up to, say roughly 20th, the result from linear delta-expansion is slightly more accurate. Then, as the order increases, the crossover occurs and at large orders, the

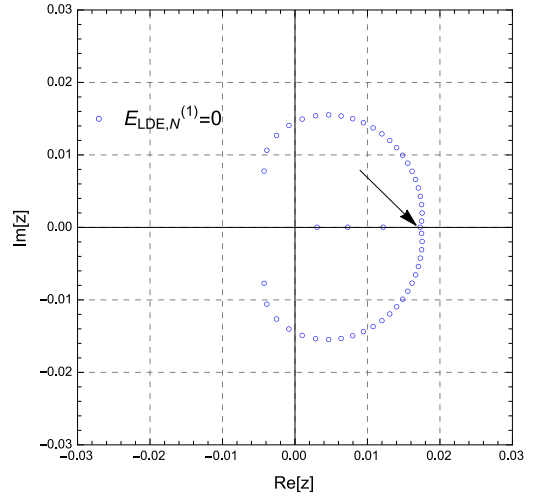


FIG. 11. The plots of zeros of the first derivative $\bar{E}_{LDE,N}^{(1)}(z)$ at $N = 50$. Here the argument z is the complex extension of λ/m^3 .

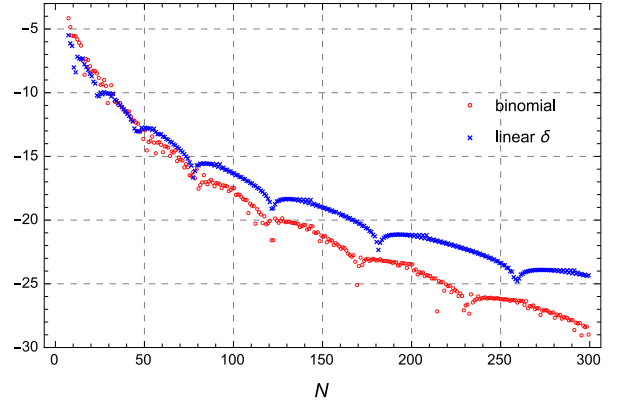


FIG. 12. The plots of $\log_{10} |E^* - \mathcal{E}|$ in the linear delta-expansion and binomial transform.

results from binomial transform become superior. Since the sequence in binomial scheme achieves higher accuracy than the sequence (which convergence is proved) from linear delta expansion, its convergence is now verified.

In the case of the anharmonic oscillator, we have used no technique special to the model. The high accuracy of the estimation comes from the analytic structure with respect to m^2 of the strong coupling series of the ground state energy [16],

$$E(m, \lambda) = \mathcal{E} \lambda^{1/3} \{1 + e_1(m^2 \lambda^{-2/3}) + e_2(m^2 \lambda^{-2/3})^2 + \dots\}. \quad (3.13)$$

Linear delta expansion eliminates lower order terms since $\{m^2(1 - \delta)\}^L = 0$ for $N \geq L$. Also in binomial method, $\mathcal{B}_N[(m^2)^L] = \binom{N}{L} t^L = 0$ since $\binom{N}{L} = \Gamma(N)/\{\Gamma(-L + 1)\Gamma(N + L + 1)\} = 0$ for $L = 1, 2, 3, \dots$. That is, both of the linear delta expansion and binomial transform methods receive an advantage from the fact that $(m^2)^n \rightarrow 0$ ($n = 1, 2, 3, \dots$) after the expansion or transformation. This is the reason why linear delta expansion and binomial transform with respect to $m^2 \lambda^{-2/3}$ yield accurate estimates unlike

TABLE III. Estimation of $\mathcal{E} = 0.667986259155777108270962016919860199430404936 \dots$ in sample orders in linear delta expansion and generalized binomial transform approaches. The exact figures in each digits are written in Roman style while figures in the last two digits including errors are written in Slanted style.

N	linear delta expansion	generalized binomial transform
10	0.6679857	0.6679837
15	0.66798630	0.6679858
20	0.667986262	0.667986268
25	0.66798625920	0.6679862579
50	0.66798625915592	0.667986259155758
100	0.66798625915577705	0.6679862591557771053
150	0.66798625915577710839	0.6679862591557771082714
200	0.66798625915577710827034	0.667986259155777108270959
250	0.667986259155777108270957	0.667986259155777108270962022
300	0.66798625915577710827096248	0.667986259155777108270962016928

the case of Laplace integral where corrections of power series in $1/t$ remain.

E. Estimation of the strong coupling coefficients

Analyticity with respect to m^2 expressed in (3.13) can be numerically confirmed by binomial transform, as we can see below: Assume that there are fractional power-like terms and let the leading one be $\text{const} \times (m^2/\lambda^{2/3})^\Delta$ ($\Delta > 0$). Then

$$\mathcal{B}[E(m, \lambda)] = \mathcal{E} \lambda^{1/3} \{1 + \text{const} \times t^{-\Delta} + \dots\}. \quad (3.14)$$

The leading correction from $t^{-\Delta}$ must then be observed in $\bar{E}_N(t, \lambda)$ if it would exist, as would be seen in the plot of ψ_0 in FIG. 1 where t^{-1} correction is active. But the numerical plot shown in FIG. 6 does not imply any power like correction. This means that the terms of fractional powers are absent in the strong coupling expansion. Thus, the expansion (3.13) is ensured even in our numerical study.

The coefficient $\alpha_k = \mathcal{E} e_k$ of the series (3.13) can be estimated in the following way: As the first example, we explain the estimation of α_1 . Setting $\lambda = 1$, consider the derivative of $E(m^2, 1)$ with respect to m^2 denoted as $\partial E(m^2, 1)/\partial m^2 := E'(m^2, 1)$. We obtain at small m^2

$$E'(m, 1) = \alpha_1 + 2\alpha_2 m^2 + 3\alpha_3 (m^2)^2 \dots, \quad (3.15)$$

and at large m^2

$$E'_N(m, 1) = \sum_{n=0}^N a_n \left(-\frac{3n-1}{2}\right) (1/m^2)^{(3n+1)/2}. \quad (3.16)$$

Then, the binomial transform eliminates corrections of integer powers of m^2 in (3.13) and may simply leave

$$\mathcal{B}[E'(m, 1)] \sim \alpha_1, \quad (3.17)$$

TABLE IV. Estimation results of coefficients $\alpha_k = \mathcal{E} e_k$ ($k = 1, 2, 3, 4, 5$) of the strong coupling expansion of the ground state energy at 250th and 300th orders. The results are expressed to the digit of order 10^{-28} the same order of correct \mathcal{E} estimated at $N = 300$ (cf. Table III).

$\alpha_1 (N = 250)$	0.1436687833808649100203190808
$\alpha_2 (N = 250)$	-0.0086275656808022791279635744
$\alpha_3 (N = 250)$	0.0008182089057563495424151582
$\alpha_4 (N = 250)$	-0.0000824292171300772199109668
$\alpha_5 (N = 250)$	0.0000080694942350409647544789
$\alpha_1 (N = 300)$	0.1436687833808649100203191272
$\alpha_2 (N = 300)$	-0.0086275656808022791279637461
$\alpha_3 (N = 300)$	0.0008182089057563495424155947
$\alpha_4 (N = 300)$	-0.0000824292171300772199118949
$\alpha_5 (N = 300)$	0.0000080694942350409647560181

at a certain region where (3.17) is expected to be recovered by $\mathcal{B}[E'_N(m, 1)]$. As in the same manner of estimating \mathcal{E} , we have carried out estimation of α_1 by substituting

$$\mathcal{B}[E'_N(m, 1)] = \sum_{n=0}^N a_n \left(-\frac{3n-1}{2}\right) \binom{N}{\frac{3n+1}{2}} t^{(3n+1)/2} \quad (3.18)$$

into $\mathcal{B}[E'(m, 1)]$ and using PMS to pick out the optimal solution for α_1 . For higher order coefficients, using the derivatives of $E(m^2, 1)$ with respective m^2 , we can estimate α_2, α_3 and so on. We tabulate the results in TABLE IV. Having compared our results at $N = 250$ and 300, we consider that the figures at $N = 250$ to 10^{-24} order are correct for α_k ($k = 1, 2, 3, 4, 5$).

Let us compare our results with those obtained by Janke and Kleinert at order $N = 251$ [15]. Their results are

$$\alpha_1^{JK} = 0.1436687833808649100203,$$

$$\begin{aligned}
\alpha_2^{JK} &= -0.008627565680802279128, \\
\alpha_3^{JK} &= 0.000818208905756349543, \\
\alpha_4^{JK} &= -0.000082429217130077221, \\
\alpha_5^{JK} &= 0.000008069494235040966.
\end{aligned} \tag{3.19}$$

As in the case of \mathcal{E} , our results for α_k ($k = 1, 2, 3, 4, 5$) are more accurate than α_k^{JK} about $2 \sim 3$ digits. We thus conclude that, as long as the order is high, the estimate of strong coupling coefficients is better in binomial transform.

F. Binomial transform with respect to λ

The computation of the ground state energy has so far been done by taking the energy as a function of m^2 . To test the flexibility of the generalized binomial transform approach, we investigate here the transform with respect to the coupling constant by taking the energy as a function of λ . To dilate the region around the strong coupling limit $\lambda = \infty$, we re-scale $\lambda = g/(1 - \delta)$ and expand the energy function in δ to the relevant order of perturbation series.

For the sake of notational simplicity, we set $m^2 = 1$. Then, the behavior of $E(1, \lambda)$ at $\lambda \gg 1$ reads from (3.13)

$$E(1, \lambda) = \mathcal{E}\lambda^{1/3}(1 + e_1\lambda^{-2/3} + e_2\lambda^{-4/3} + \dots), \tag{3.20}$$

and at $\lambda \ll 1$

$$E_N(1, \lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \dots + a_N\lambda^N. \tag{3.21}$$

We investigate the computation of \mathcal{E} in most part without using the values of the exponents of the corrections to the asymptotic term. We thus start with

$$E(1, \lambda) = \mathcal{E}(\lambda^{1/3} + e_1\lambda^{-\theta_1} + e_2\lambda^{-\theta_2} + \dots). \tag{3.22}$$

Such a supposed situation is interesting to establish a benchmark of our approach to capture the leading term in the fractional power expansion from divergent series around a certain accessible point ($\lambda = 0$ here).

The basic result needed for the transform is then

$$\mathcal{B}[\lambda^s] = \binom{N}{s} g^s. \tag{3.23}$$

Though the transform improves the simulation task of $\bar{E}(1, g)$ via $\bar{E}_N(1, g)$, to achieve a good accuracy, we need to reduce the correction. So we must somehow estimate first the values of the exponents θ_i up to, say, a first few i . It has turned out, though, that higher order θ_i is difficult to estimate precisely in stable and systematic manner. Here we suffice ourselves with the reduction of the first order correction by the estimation of $\theta_1 = 1/3$ and make use of the result for the estimation of \mathcal{E} .

We start with noting that, since the leading order correction has the exponent $1/3$ known on dimensional grounds, the leading term can be eliminated in the following combination,

$$\begin{aligned}
\bar{E} - 3\bar{E}^{(1)} &= \mathcal{E} \left[e_1 \binom{N}{\theta_1} (1 + 3\theta_1)g^{-\theta_1} \right. \\
&\quad \left. + e_3 \binom{N}{\theta_2} (1 + 3\theta_3)g^{-\theta_3} + \dots \right],
\end{aligned} \tag{3.24}$$

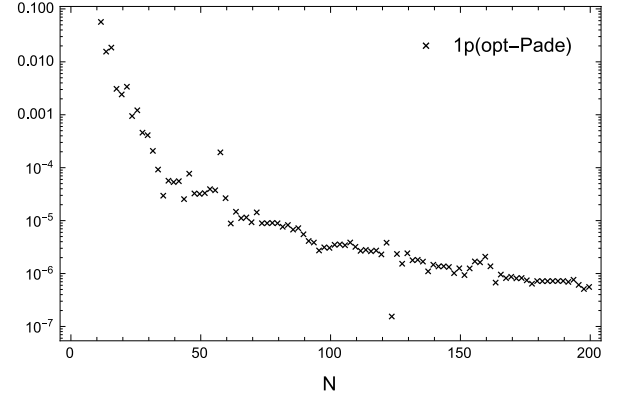


FIG. 13. The logarithmic plots of $|\theta_1 - 1/3|$ up to 200th order.

with $\bar{E}^{(1)} = (\partial/\partial \log g)\bar{E}(1, g)$ understood. Here we have used the fact that $\mathcal{B}_N[\lambda^{-\theta_2}] = \mathcal{B}_N[\lambda^{-1}] = 0$. Equally, all the terms with negative integer powers are eliminated by the binomial transform. To avoid notational complexity, we re-parametrize exponents as $\theta_1 (= 1/3)$, $\theta_2 (= 5/3)$, $\theta_3 (= 7/3)$, $\theta_4 (= 11/3)$ etc. Then in general, it holds that

$$\prod_{i=0}^L \left[1 + \frac{1}{\theta_i} \frac{\partial}{\partial \log g} \right] \bar{E} = \text{const.} \times g^{-\theta_{L+1}} + O(g^{-\theta_{L+2}}), \tag{3.25}$$

where $\theta_0 = -1/3$. Now, taking the following quotient and expanding it in $1/g$, we find

$$Q_L = \frac{\prod_{i=0}^L \left[1 + \frac{1}{\theta_i} \frac{\partial}{\partial \log g} \right] E^{(1)}}{\prod_{i=0}^L \left[1 + \frac{1}{\theta_i} \frac{\partial}{\partial \log g} \right] E} = -\theta_{L+1} + \dots, \tag{3.26}$$

where the dots means the correction of order $O(g^{-\theta_{L+2} + \theta_{L+1}})$. We here concern with the case $L = 0$, giving at large g ,

$$Q_0 = \frac{\left[1 - 3 \frac{\partial}{\partial \log g} \right] E^{(1)}}{\left[1 - 3 \frac{\partial}{\partial \log g} \right] E} = -\theta_1 + \dots, \tag{3.27}$$

The quotient Q_0 is a divergent series in g which we denote as $Q_{0,N}$, where \bar{E}_N and $\bar{E}_N^{(1)}$ are used in the places of \bar{E} and $\bar{E}^{(1)}$. The function $Q_{0,N}$ thus defined looks like the Padé-type rational function but it is actually not. Hence by first expanding $Q_{0,N} = [1 - 3 \frac{\partial}{\partial \log g}] E_N^{(1)} / [1 - 3 \frac{\partial}{\partial \log g}] E_N$ in g , we construct diagonal Padé approximants to circumvent the zero-convergence-radius difficulty. We then take the $g \rightarrow \infty$ limit of $Q_{0,N}[N/2, N/2]$ as the estimate of θ_1 . That is, from (3.27),

$$\lim_{g \rightarrow \infty} Q_{0,N}[N/2, N/2] = -\theta_1. \tag{3.28}$$

The result is plotted in FIG. 13 with the label "1p(opt-Pade)". The sequence does not exhibit clear shape and the convergence issue is not definitive. However, the estimation of θ_1

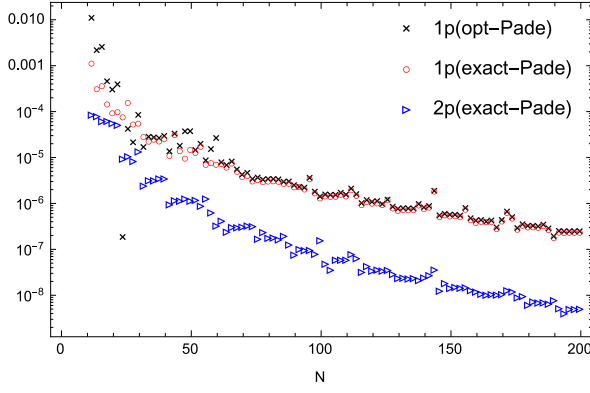


FIG. 14. The logarithmic plots of $|E^* - \mathcal{E}|$ up to 200th order in the three methods: The "1p(opt-Pade)" shows the result from the first order reduction via $[1 + 3(\partial/\partial \log g)]\bar{E}/\{(-N/3)g^{1/3}\}$ with the order dependent optimal θ_1 . The "1p(exact-Pade)" sequence shows the result via $[1 + 3(\partial/\partial \log g)]\bar{E}/\{(-N/3)g^{1/3}\}$ with exact θ_1 and the "2p(exact-Pade)" via $[1 + 3/5(\partial/\partial \log g)][1 + 3(\partial/\partial \log g)]\bar{E}/\{(-N/3)g^{1/3}\}$ with exact θ_1 and θ_2 .

is accurate enough and we make use of the estimate at order N to the estimate of \mathcal{E} at the same order in the following manner.

One technical point to be paid attention is that the asymptotic of \bar{E} in $g \rightarrow \infty$ is $\mathcal{E}(-N/3)g^{1/3}$ which is g dependent. The ground state energy may be estimated in this case by dividing \bar{E} by $(-N/3)g^{1/3}$. To improve the accuracy, however, we again use first order reduced function $[1 + (1/\theta_1)\partial/\partial \log g]\bar{E}$. We thus study the combination $(\bar{E} + (1/\theta_1)\bar{E}^{(1)})/\{(-N/3)g^{1/3}\}$ which tends to \mathcal{E} in the large g limit,

$$\lim_{g \rightarrow \infty} \frac{\bar{E} + \frac{1}{\theta_1}\bar{E}^{(1)}}{(-N/3)g^{1/3}} = \mathcal{E}. \quad (3.29)$$

Now, in the use of \bar{E}_N and $\bar{E}_N^{(1)}$, the above limit does not hold even when we employ Padé approximants of any $[\rho/\tau]$ element in the numerator $\bar{E}_N + \frac{1}{\theta_1}\bar{E}_N^{(1)}$ of (3.29). This is reasonable, since beyond narrow effective region of the series expansion Padé approximants take extrapolation affected by the highest orders g^ρ and g^τ of the numerator and the denominator, giving the behavior $(\bar{E} + \frac{1}{\theta_1}\bar{E}^{(1)})[\rho/\tau] \sim \text{const.} \times g^{\rho-\tau}$ which exponent cannot agree with $1/3$, the power of $g^{1/3}$. The expected asymptotic behavior $\sim g^{1/3}$ should occur at a certain region where g is not so large. The reliable region is indicated by the plateau in the combination (3.29) and it serves an optimal estimation point of \mathcal{E} under PMS. Due to the smallness of the exponent $1/3$ which should be recovered by Padé approximants, best choice is the diagonal one. Then we find that the optimal point occurs at either the extremum or the reflection points. We also keep the estimate when poles on the positive real axis exist.

The result is plotted in FIG. 14 by the black cross labeled by "1p(opt-Pade)". In FIG. 14, we have also plotted, for the comparison with the same optimization procedure, the result at the

first and second order reductions of the corrections with exact exponents. The "1p(exact-Pade)" labelled sequence shows the result via the quotient $[1 + 3(\partial/\partial \log g)]\bar{E}/\{(-N/3)g^{1/3}\}$ and the "2p(exact-Pade)" via the quotient $[1 + 3/5(\partial/\partial \log g)][1 + 3(\partial/\partial \log g)]\bar{E}/\{(-N/3)g^{1/3}\}$. We now see that the optimal solution of θ_1 provides almost same accuracy with the use of the exact value of θ_1 . The reduced function to the second order brought more accuracy as it would.

Though not so clear, all the above three sequences to $N = 200$ appear to converge from the plots. One evidence of the convergence of the sequences comes from the behaviors of estimation point $t = t^*$. All three sequences of t^* show gradual increase with the order. This means that the approximate region is indeed extrapolated to larger g and, consequently, the convergence becomes quite conceivable due to the disappearance of the binomial coefficient of $g^{-\theta_i}$ ($i = 1, 2, 3, \dots$), $(-N/3)$ in the $N \rightarrow \infty$ limit.

The reduction of the correction lead by the left-hand-side of (3.25) gives further accurate estimates as the order of reduction L is increased. By using exact values of θ_i to more many orders, we obtain

$$L = 10 : 0.66798625915577710858725991, \quad (3.30)$$

$$L = 20 : 0.66798625915577710827111996, \quad (3.31)$$

$$L = 30 : 0.66798625915577710827096268, \quad (3.32)$$

at $N = 250$. The results are respectively exact to 10^{-18} , 10^{-20} and 10^{-24} orders. The last result achieved the accuracy with the almost same order with the one in linear delta expansion. The case $L = 40$ gives less accurate result at $N = 250$ than the case $L = 30$. But we confirmed that at $N = 300$ the accuracy exceeded than the case $L = 30$: $\text{error} = 4.7 \times 10^{-25}$ ($L = 30$) and $\text{error} = 8.5 \times 10^{-30}$ ($L = 40$). The effect of the incorporated exponents shows up at relatively larger orders.

IV. CONCLUDING REMARKS

We have explored the generalized binomial transform in the application to a Laplace integral and the quantum anharmonic oscillator in the perturbative framework.

In the Laplace integral function, the use of the technique to subtract corrections in assumed power series have found to give accurate estimation of the function in the limit $\lim_{M \rightarrow 0} f(0)$ from $1/M$ divergent expansion. The extrapolation to the infinite order by a simple fitting predicts slightly different value, which is caused by the nature of divergent series. To go beyond the zero-convergence-radius difficulty, we found that diagonal Padé approximants of transformed series provided excellent estimation at finite order and solid evidence of convergence to the exact limit.

In the case of the anharmonic oscillator, the ground state energy computation is successful in the naive transformed function by detecting the optimal stationary or almost stationary points among many candidates under PMS. The accuracy of estimate is periodically improving as the order grows to

$N = 300$ and the sequence of estimate is found by the comparison with that in the linear delta expansion to converge to the exact value. Also the coefficients in the strong coupling expansion could be estimated with the same level of accuracy.

From the point of view that the ground state energy is considered as a function of the coupling constant, we also studied the binomial transform approach to the computation of \mathcal{E} . Explicit reduction provided θ_i is not known was carried out in the case of one-parameter (θ_1) reduction, and a good estimation was shown to be possible.

In these three estimation tasks, we found that when the transformed function has power like correction in the target region of the argument, nature of divergent series usually forbids the exact convergence of the sequence of estimates from

transformed polynomial, though the improvement by the reduction of the correction to the limit produces accurate results. The problem has been resolved by the Padé approximant method by which the successful extrapolation beyond original narrow effective region was achieved. Note that, in the study of the anharmonic oscillator based on the binomial transform with respect to λ , reduction of the first order correction was crucial for the estimation by the Padé approximants.

In various physical models, the combined use of the correction-reduction and Padé approximants may become useful tools for dealing with the divergent series of a given function, usually accessible in the perturbative side, for the quantitative computation of the function in the opposite target region, in particular when the knowledge in the target region is less informed.

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$$H(\delta; a, b_n) = \frac{1}{2} \left(\frac{\partial \phi}{\partial q} \right)^2 + \frac{m^2}{2} (1 - \delta) \{ 1 + (a - 1) \delta + \sum_{n=1} b_n \delta^{n+1} \} \phi^2 + \delta \lambda \phi^4,$$
and the physical interpretation is obscured.
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